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ON DIVIDED COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring with identity having total quotient ring T. A prime ideal P of R is called divided if P is comparable to every principal ideal of R. If every prime ideal of R is divided, then R is called a divided ring. If P is a nonprincipal divided prime, then $P^{-1} = \{ x \in T : xP \subset P \}$ is a ring. We show that if R is an atomic domain and divided, then the Krull dimension of $R \leq 1$. Also, we show that if a finitely generated prime ideal containing a nonzerodivisor of a ring R is divided, then it is maximal and R is quasilocal.

INTRODUCTION

Through out this paper, R denotes a commutative ring with 1 and T denotes the total quotient ring of R. Given a ring R, then Z(R) denotes the set of zerodivisors of R, and N denotes the set of nonunits of R. D. Dobbs in [8] studied divided domains. Our main purpose is to generalize the study of divided domains to the context of arbitrary rings where possibly Z(R) is nonzero. Our definition of divided rings is the same as that one given in [8] for integral domains.

We start with the following definitions :

Definition. A prime ideal P of a ring R is called divided if P is comparable to every principal ideal of R. If every prime ideal of R is divided, then R is called a divided ring.

Definition. Recall from [6], a prime ideal P of R is called strongly prime if aP and bR are comparable for every $a,b \in R$. If every prime ideal of a ring R is strongly prime, then R is called a pseudo-valuation ring (PVR).

The first part of the following result is clear by the definition of divided rings, and the second part is also clear by [6, Lemma 1(a)] where it was shown that if a prime ideal P of R is strongly prime, then it is comparable to every principal ideal of R and therefore it is divided.

Proposition 1. (a). If R is a divided ring, then the prime ideals of R are linearly ordered and therefore R is quasi-local. **(b).** If R is a PVR, then R is a divided ring.

In [5, Proposition 2] we gave several characterizations of divided domains. In view of the proof of [5, Proposition 2], we see that these characterizations still valid for an arbitrary ring R. Thus, we state them here without proof.

Proposition 2. The following statements are equivalent for a ring R.

(1) R is a divided ring.

(2) For every pair of proper ideals I, J of R, I and Rad(J) are comparable, where Rad(J) denotes the radical of J.

(3) For every $a,b \in R$, the ideals (a) and Rad((b)) are comparable.

(4) For every $a,b \in R$, either $a \mid b$ or $b \mid a^n$ for some $n \geq 1$.

In light of Proposition 2(4), we have the following result.

Corollary 3. Any homomorphic image of a divided ring is divided. In particular, if R is divided and I is an ideal of R, then R/I is divided.

The following result is a generalization of [8, Lemma 2.2 (a)]. Our proof is different than that given in [8].

Proposition 4. Any localization of a divided ring is divided.

Proof. Let S be a nonzero multiplicatively closed subset of R and $x, y \in R_S$. Then $x = a/s$ and $y = b/s$ for some $s \in S$ and $a, b \in R$. Since R is divided, $a|b$ or $b|a^n$ for some $n \geq 1$ by Proposition 2(4). Hence, $b = ca$ or $a^n = gb$ for some $c, g \in R$. Thus, $b/s = c(a/s)$ or $a^n/s^n = (g/s^{n-1})(b/s)$. Thus, $x|y$ or $y|x^n$. Therefore, R is divided by Proposition 2(4). ∎

In [7, Theorem 1], we proved that R is a PVR if and only if for every $a, b \in R$, bR and aN are comparable. The following result is an analog of this fact.

Proposition 5. The following statements are equivalent for a ring R.

(1) R is divided.

(2) For every $a, b \in R$, there is an $n \geq 1$ such that bR and a^nN are comparable.

Proof. (1)$\Longrightarrow$(2). Suppose that $bR \not\subset a^mN$ for every $m \geq 1$. Then either a and b are associative in R or a does not divide b in R. If a and b are associative, then

$b|a$ and therefore $aN \subset bR$. If a does not divide b in R, then $b|a^n$ for some $n \geq 1$ by Proposition 2(4) and hence $a^nN \subset bR$. **(2) $\Longrightarrow$ (1).** Let $a,b \in R$. By proposition 2(4) we need show that either $a|b$ or $b|a^n$ for some $n \geq 1$. Now, if $bR \subset a^nN$ for some $n \geq 1$, then $a|b$. If $a^mN \subset bR$ for some $m \geq 1$, then $b|a^{m+1}$. Thus, R is a divided ringt. ∎

A consequence of the above result is the following corollary.

Corollary 6. Let R be a quasilocal ring with the maximal ideal M. The following statements are equivalent.

(1) R is divided.

(2) For every $a,b \in R$, there is an $n \geq 1$ such that bR and a^nM are comparable.

Recall that if I is an ideal of R, then $I^{-1} = \{ x \in T : xI \subset R \}$ and $I:I = \{ x \in T : xI \subset I \}$. We leave the proof of the following lemma to the reader.

Lemma 7. Let I be a nonprincipal ideal of R. Then $xI \subset N$ for every $x \in I^{-1}$.

The following lemma is needed in the next result.

Lemma 8. Let P be a divided prime ideal of R containing a nonzerodivisor of R. Then $Z(R) \subset P$.

Proof. Let $s \in P$ be a nonzerodivisor of R. Suppose that there is a $z \in Z(R)\backslash P$. Since P is divided, $P \subset (z)$ and in particular $z|s$ which is impossible. Thus, $Z(R) \subset P$. ∎

The following result is a generalization of the first part in [4, Proposition 6].

Proposition 9. Let P be a nonprincipal divided prime ideal of R. Then $P^{-1} = P:P$ is a ring.

Proof. Suppose there is an $x \in P^{-1}\backslash R$. Write $x = a/b$ for some $a \in R$ and a nonzerodivisor $b \in R$. Suppose that for some $p \in P$, $(a/b)p = c \in R\backslash P$. Then $ap = bc$ in R. Hence, $(a/b)(p/c) = 1$ in T. Since P is prime and $c \in R\backslash P$, $b \in P$. Hence, $Z(R) \subset P$ by Lemma 8. Thus, c is a nonzerodivisor of R. Since P is divided and $c \in R\backslash P$, $p/c \in P$. But $(a/b)(p/c) = 1$ which is a contradiction by Lemma 7. Thus, $P^{-1} = P:P$ is a ring. ∎

The following is a generalization of [4, Proposition 7].

Proposition 10. Let I be a proper ideal of R containing a nonzerodivisor of R. The following statements are equivalent.

(1) I is a nonprincipal divided prime ideal.

(2) I^{-1} is a ring and I is comparable to every principal ideal of R.

Proof. **(1) $\Longrightarrow$ (2).** This is clear by Proposition 9 and the definition of divided prime. **(2) $\Longrightarrow$ (1).** Since I Contains a nonzerodivisor of R and it is comparable to every principal ideal of R, we see that $Z(R) \subset I$. Since I^{-1} is a ring and I contains a nonzerodivisor of R, I is nonprincipal. For, if I is principal, then $I = (s)$ for some nonzerodivisor $s \in R$. Hence, $1/s \in I^{-1}$. Since I^{-1} is a ring, $1/s^2 \in I^{-1}$. But $(1/s^2)s = 1/s \notin R$, a contradiction. Now, we show that I is prime. Let $S = R\backslash I$ and $x,y \in S$. Since $Z(R) \subset I$, neither x nor y is a zerodivisor of R. Since I is comparable to every principal ideal of R, $1/x$ and $1/y$ are in I^{-1}. Since I^{-1} is a ring, $(1/x)(1/y) = 1/xy \in I^{-1}$. Since I is nonprincipal and $1/xy \in I^{-1}$, $xy \in S$. Thus, S is a multiplicatively closed subset of R and therefore I is prime ∎

The following example shows that the hypothesis that I contains a nonzerodivisor of R is crucial.

Example 11. Let $R = Z_8$ and $I = (2)$. Then $I^{-1} = R$ is a ring and I is divided but I is principal.

In view of Example 11, we have the following result.

Proposition 12. Let I be a proper ideal of R such that $Z(R) \subset I$. If I^{-1} is a ring and I is comparable to every principal ideal of R, then I is prime.

Proof. To show that I is prime, see the argument given in the proof of Proposition 10. ∎

The following example shows that the hypothesis $Z(R) \subset I$ is crucial in the above Proposition.

Example 13. Let $R = Z_8$ and $I = (4)$. Then $I^{-1} = R$ is a ring and I is comparable to every principal ideal of R but I is not prime.

The first part of the following lemma is taken from [5, Theorem 1].

Lemma 14. (a). The prime ideals of a ring R are linearly ordered if and only if the radical of every proper principal ideal of R is prime if and only if for every $a,b \in R$, either $a|b^n$ or $b|a^m$ for every $n,m \geq 1$. **(b).** If $a,b \in R$, then $Rad((a)) = Rad((b))$ if and only if there are $n,m \geq 1$ such that $a|b^n$ and $b|a^m$.

Proof. (b). Just observe that $Rad((a)) = Rad((b))$ iff $a \in$

D. Dobbs in [9, Proposition 2.2 (a)] proved that if P is a divided prime ideal of a domain R, then P^n is a P-primary ideal of R, for every $n \geq 1$. The following is a generalization of this fact.

Proposition 17. Let P be a divided prime ideal of R such that $Z(R) \subset P$. Then P^n is P-primary, for every $n \geq 1$.

Proof. We show that if $a, b \in R$ satisfy $ab \in P^n$ and $a \notin \text{Rad}(P^n) = P$, then $b \in P^n$. Consider an element of the form $y = p_1 p_2 \ldots p_n$ in P^n where the p_i's are in P. To show $b \in P^n$, it suffices to show that $y/a \in P^n$, since b is a finite sum of element of the form of y. Since $Z(R) \subset P$ and $a \notin P$ and P is divided, a is a nonzerodivisor of R and $p/a \in P$ for every $p \in P$. Thus, $y/a = (p_1/a)p_2 \ldots p_n \in P^n$. Hence, $b \in P^n$. Therefore, P^n is P-primary. ∎

The following example shows that the hypothesis $Z(R) \subset P$ in the above Proposition is crucial.

Example 18. Let $V = Z_{(2)} + XQ[[X]]$, a two dimensional valuation domain with prime ideals $(0) \subset P = XQ[[X]] \subset M = 2Z_{(2)} + XQ[[X]]$. Let $R = V/X^2V$. Then R is a PVR (see [6, Example 10(b)]) with prime ideals $G = P/X^2V$ and $Z(R) = N = M/X^2V$. Then G is divided. Now, $(2 + X^2V)(X^2/2 + X^2V) \in G^3 = [P^3 + X^2V] / X^2V = 0$ in R. But neither $2 + X^2V \in \text{Rad}(G^3) = G$ nor $X^2/2 + X^2V \in G^3$ since $1/2 \notin X^2V$. Hence, G^3 is not G-primary of R.

Proposition 19. Let R be an atomic domain. Then R is divided if and only if R is quasilocal of Krull dimension 1.

Proof. Suppose that R is divided with maximal ideal M. Suppose that there is a nonzero prime ideal P of R such

Rad((b)) and $b \in$ Rad((a)) iff there are $n, m \geq 1$ such that $a|b^n$ and $b|a^m$. ■

Recall that a ring B is called an overring of R if $R \subset B \subset T$. A prime ideal P of R contains a nonzerodivisor element of R is called a minimal regular prime ideal of R if whenever $Q \subsetneq P$ for some prime ideal Q of R, then $Q \subset Z(R)$.

Proposition 15. Suppose that the prime ideals of a ring R are linearly ordered, and B is an overring of R containing an element of the form 1/s for some nonunit nonzerodivisor $s \in R$. Furthermore, suppose that Rad((s)) is a minimal regular prime ideal of R, then B = T. In particular, if R is divided, then B = T is divided.

Proof. To show that B = T, it suffices to show that $1/d \in B$ for every nonzerodivisor $d \in R$. Let d be a nonzerodivisor of R. We consider two cases : <u>case 1</u>. Suppose that $d \in R\backslash$Rad((s)). Then $d|s^n$ for some $n \geq 1$ by Lemma 14 (a). Hence, $s^n = dk$ for some $k \in R$. Thus, $k/s^n = 1/d$ in T. Since $1/s \in B$, $k/s^n = 1/b \in B$. <u>Case 2</u>. Suppose that $d \in$ Rad((s)). Since Rad((s)) is a minimal prime ideal of R and Rad((d)) is prime by Lemma 14 (a), Rad((s)) = Rad((d)). Hence, $d|s^n$ for some $n \geq 1$ by Lemma 14 (b). Now, a similar argument as in case 1, we conclude that $1/d \in B$. Thus, B = T. The remaining part is clear by Proposition 4. ■

In light of the above Proposition, we have the following.

Corollary 16. Let R be a quasilocal ring of a Krull dimension 1 containing a nonunit nonzerodivisor element. Then T is the only overring of R containing an element of the form 1/s for some nonunit nonzerodivisor $s \in R$.

that $P \not\subset M$. Then there are atoms a,b of R such that $a \in P$ and $b \in M\backslash P$. Since P is divided, $b|a$ which is a contradiction. The converse is clear. ∎

The following lemma is well-known. See for example [10, Theorem 15]. We state it here without proof.

Lemma 20. Let s be a nonunit nonzerodivisor element of R. Then $1/s$ is never integral over R.

It is easy to see that if P is a divided prime ideal of R, then $P \subset J(R)$ where $J(R)$ is the Jacobson radical of R. In the following result, we show that if a finitely generated prime ideal containing a nonzerodivisor of R is divided, then P is maximal and therefore R is quasilocal.

Proposition 21. Let P be a finitely generated prime ideal containing a nonzerodivisor of R. If P is divided, then P is maximal and therefore R is quasilocal.

Proof. Deny. Then there is an $s \in N\backslash P$. Since P contains a nonzerodivisor of R, $Z(R) \subset P$ by Lemma 8. Hence, s is a nonzerodivisor of R. Since P is prime and divided, $(1/s)P \subset P$. Since P contains a nonzerodivisor of R, the annihlator of P in T is 0. Hence, by [10, Theorem 12], $1/s$ is integral over R. A contradiction by Lemma 20. ∎

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